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LETTER TO THE EDITOR

On the Thomas equation

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Abstract. The Thomas equation has the Lie–Bäcklund algebra recurrence operator of the first order, the infinite set of the many-parameter Bäcklund autotransformations and no reduction to the Painlevé transcendents. The Burgers equation has the same properties which are probably the general indicators of the hidden linearity.

In 1944 Thomas proposed [1] the non-linear equation

$$\varphi_{xt} + \alpha\varphi_x + \beta\varphi_t + \varphi_x\varphi_t = 0 \tag{1}$$

(α and β are arbitrary constants) describing a certain chemical reaction and found the linearising transformation

$$\varphi = -\beta x - \alpha t + \ln \omega \tag{2}$$

that transforms (1) into the well-studied linear hyperbolic equation

$$\omega_{xt} = \alpha\beta\omega. \tag{3}$$

Equation (1) was described in [2] as the Thomas equation (TE) together with its linearising transformation (2). Recently, the TE was investigated intensively in the series of works [3–5]. There, for some reason the TE was called the Thompson equation, the linearisation (2), (3) was not mentioned at all, and the TE was investigated as an essentially non-linear equation by means of all the arsenal of powerful modern 'non-linear' methods such as the higher symmetries and the Lax pair [3], the reduction to ODE and the Painlevé property [4], the prolongation algebra and the Bäcklund transformation [5]. So on the one hand the TE has the hidden linearity, i.e. it can be linearised exactly, and on the other hand it is a well studied completely integrable equation. That is why the TE is very useful for answering the following rather general question: what are the properties of a certain non-linear equation that indicate the possibility of linearising that equation exactly? Taking the results of [3–5] as a basis (after some corrections and additions) we find three possible indicators of the hidden linearity. They are not some special properties of the TE only. To demonstrate it, we consider the well known Burgers equation (BE) [6]

$$u_t = u_{xx} - uu_x \tag{4}$$

which is linearised into

$$v_t = v_{xx} \tag{5}$$

via the Cole-Hopf transformation [7, 8]

$$u = -2v_x/v. \tag{6}$$

In [3] the higher symmetries of the TE were investigated. Unfortunately, the Lie-Bäcklund algebra [9] basis of the TE was wrongly calculated there, the whole of its t part and all the even-order elements of its x part having been left out. As a result of the mistake, the recurrence operator of the TE Lie-Bäcklund algebra (LBA) was found to be of second order which does not distinguish the TE from such equations as the sine-Gordon equation (SG) and the Korteweg-de Vries equation (KdV). Calculating the basis of the TE LBA correctly, we find it to be $1, \varphi_x, \varphi_{xx}+\varphi_x^2, \varphi_{xxx}+3\varphi_x\varphi_{xx}+\varphi_x^3, \ldots, \varphi_t, \varphi_{tt}+\varphi_t^2, \varphi_{ttt}+3\varphi_t\varphi_{tt}+\varphi_t^3, \ldots$. The basis evidently has two first-order recurrence operators $L=\partial_x+\varphi_x$, $M=\partial_t+\varphi_t$, and the whole of it can be constructed from any of its elements via these operators. The first order of the recurrence operators distinctly distinguishes the TE from other non-linear completely integrable PDE, whose exact linearisations are unknown as yet or probably do not exist at all, the LBA of the KdV and the SG have second-order recurrence operators, the LBA of the Sawada-Kotera equation has sixth order [9], etc. This property of the TE is not a chance phenomenon and can be easily explained in terms of its hidden linearity. Indeed, the LBA of the linear equation (3) has the basis $\omega, \omega_x, \omega_{xx}, \omega_{xxx}, \ldots, \omega_t, \omega_{tt}, \omega_{ttt}, \ldots$ with first-order recurrence operators ∂_x and ∂_t, while the equivalence transformation (2) does not change the order of the recurrence operators. The same is true for the BE (4). The basis of its LBA $u_x, u_{xx}-uu_x, u_{xxx}-\frac{3}{2}uu_{xx}-\frac{3}{2}u_x^2+\frac{3}{4}u^2u_x, \ldots$ has the recurrence operator $N=\partial_x-\frac{1}{2}u-\frac{1}{2}u_x\partial_x^{-1}$ [9] of first order too. This is also attributed to the first order of the recurrence operator ∂_x of the linear equation (5) [9]. Since any linear constant-coefficients equation has a recurrence operator of first order, the wisest course would be to assume that any non-linear exactly linearisable PDE has a recurrence operator of first order too.

The exact reductions of the TE (1) to ODE were investigated in [4]. The ODE have turned out to be some reducible Painlevé equations. It should be noticed that the exact reductions of the TE do not generate the Painlevé transcendents, while the KdV, the SG, the Boussinesq equation and some other completely integrable PDE [10] (whose exact linearisations are unknown as yet or probably do not exist at all) do generate them. The Painlevé transcendents [11] are known to be irreducible equations; they cannot be reduced to some first-order equations or solved in terms of elliptic functions or transformed into some linear ODE (maybe of a higher order). To a certain extent, the Painlevé transcendents inherit essentially non-linear structures of those non-linear PDE which generate them via exact reductions. On the contrary, if an ODE is generated by an exactly linearisable PDE, it must inherit the hidden linear structure of the PDE. Indeed, both the TE [4] and the BE [10] give Painlevé equations reducible to the Riccati equation, and the latter transforms into the linear Schrödinger equation. Therefore, if some non-linear PDE has the Painlevé property [10] but does not generate Painlevé transcendents, it probably can be linearised exactly.

In [5] the prolongation structure of the TE was investigated, and the Bäcklund autotransformation (BAT), the one-soliton solution and the soliton superposition law were found. Nevertheless, the form of the prolongation algebra, the 2×2 linear problem, BAT, soliton and non-linear superposition law do not indicate that the equation under consideration can be easily linearised. The corresponding results for, e.g., the SG have a similar form. In order to detect one more indicator of the hidden linearity, let us use the following particular property of any linear constant-coefficients equation, namely that a derivative of any solution of the equation is again a solution of the equation. We can construct infinitely many BAT of the linear equation (3) (ξ_k are arbitrary constants, n is an arbitrary integer):

$$\omega' = \sum_{k=0}^{n} \xi_k \frac{\partial^k \omega}{\partial x^k} \tag{7}$$

and transform them via (2) into

$$\varphi' = -\beta x + \ln\left(\sum_{k=0}^{n} \xi_k \frac{\partial^k}{\partial x^k} \mathrm{e}^{\varphi+\beta x}\right). \tag{8}$$

Now we get the infinite sequence of the many-parameter BAT of the TE:

$$\varphi' = \varphi + \ln \xi_0$$

$$\varphi' = \varphi + \ln[(\xi_0 + \beta\xi_1) + \xi_1\varphi_x]$$

$$\varphi' = \varphi + \ln[(\xi_0 + \beta\xi_1 + \beta^2\xi_2) + (\xi_1 + 2\beta\xi_2)\varphi_x + \xi_2(\varphi_{xx} + \varphi_x^2)]$$

etc. The same is true for the BE (4). Taking BAT of (5) in the form

$$v' = \sum_{k=0}^{n} \xi_k \frac{\partial^k v}{\partial x^k} \tag{9}$$

we get from (6)

$$u' = -2 \sum_{k=0}^{n} \xi_k \frac{\partial^{k+1} v}{\partial x^{k+1}} \left(\sum_{k=0}^{n} \xi_k \frac{\partial^k v}{\partial x^k}\right)^{-1}. \tag{10}$$

For any n, v can be excluded from (10) via (6) and its prolongations ($v_x = (-\frac{1}{2}u)v$, $v_{xx} = (-\frac{1}{2}u_x + \frac{1}{4}u^2)v, \ldots$), and we have the infinite sequence of the many-parameter BAT of the BE:

$$u' = -2u_x/(u+\kappa) + u \tag{11}$$

where $\kappa = -2\xi_0/\xi_1$ (this BAT of the BE was mentioned in [10]) and

$$u' = -2[\ln(u_x - \tfrac{1}{2}u^2 + \lambda u + \mu)]_x + u \tag{12}$$

where $\lambda = \xi_1/\xi_2$, $\mu = -2\xi_0/\xi_2$, and so on; for every n there exists the n-parameter BAT of the BE. The existence of the infinite sets of the BAT for the TE and the BE is based on their hidden linearities and distinguishes them from other non-linear completely integrable equations, of which neither exact linearisations nor infinite sets of BAT are known as yet.

It should be noted that the TE and the BE are linearised via essentially different transformations. The transformation (2) is a point transformation involving only the dependent and independent variables, while the Cole–Hopf transformation (6) involves the derivative and (like the well known Miura transformation) must be classified as a Lie–Bäcklund transformation [9]. Despite this difference, the TE and the BE have three properties in common based on their hidden linearities. At the same time, other completely integrable equations, whose exact linearisations are unknown as yet or probably do not exist at all, do not have these properties. The foregoing arguments make it possible to formulate the following conjecture: a non-linear PDE has a hidden linearity, i.e. it can be transformed into some linear constant-coefficients equation, if (i) its LBA has a recurrence operator of the first order, (ii) the PDE has the Painlevé property [10] but no exact reduction to the Painlevé transcendents, and (iii) the PDE has an infinite set of many-parameter BAT.

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